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Error analysis in some Gauss–Turán–Radau and Gauss–Turán–Lobatto quadratures for analytic functions☆

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Abstract

We consider the generalized Gauss–Turán quadrature formulae of Radau and Lobatto type for approximating $\int_{-1}^{1} f(t)w(t)\,\mathrm{d}t$. The aim of this paper is to analyze the remainder term in the case when f is an analytic function in some region of the complex plane containing the interval $[-1,1]$ in its interior. The remainder term is presented in the form of a contour integral over confocal ellipses (cf. SIAM J. Numer. Anal. 80 (1983) 1170). Sufficient conditions on the convergence for some of such quadratures, associated with the generalized Chebyshev weight functions, are found. Using some ideas from Hunter (BIT 35 (1995) 64) we obtain new estimates of the remainder term, which are very exact. Some numerical results and illustrations are shown.

MSC: primary 41A55; secondary 65D30; 65D32

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1. Introduction

Let w be a nonnegative weight function on the interval $[-1,1]$. For $s,p,q \in \mathbb{N}_0$, we study interpolatory quadrature formulae of the form

$$I(f;w) := \int_{-1}^{1} f(t)w(t)\,\mathrm{d}t = Q_{n,s}^{(p,q)}(f) + R_{n,s}^{(p,q)}(f) \tag{1.1}$$

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with the maximal degree of precision, where

$$Q_{n,s}^{(p,q)}(f) := F_{n,s}^{(p,q)}(f) + G_{n,s}(f) \tag{1.2}$$

and

$$F_{n,s}^{(p,q)}(f) := \sum_{i=0}^{p-1} \alpha_i f^{(i)}(-1) + \sum_{i=0}^{q-1} \beta_i f^{(i)}(1), \tag{1.3}$$

$$G_{n,s}(f) := \sum_{v=1}^{n} \sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v) \tag{1.4}$$

with the nodes $\tau_v \in (-1,1)$, $v=1,\dots,n$. Let $\mathscr{P}_m$ be the set of all algebraic polynomials of degree at most m.

If $p,q>0$ the quadrature formula (1.2) is known as the *Gauss–Turán–Lobatto* formula. If only one of p and q is equal to zero, then the corresponding formula is known as the *Gauss–Turán–Radau* type formula. Then, one of sums on the right-hand side in (1.3) vanishes. Of course, when $p=q=0$, formula (1.2) reduces to the well-known *Gauss–Turán* quadrature formula [21]

$$\int_{-1}^{1} f(t)w(t)\,\mathrm{d}t = G_{n,s}(f) + R_{n,s}^{(0,0)}(f), \tag{1.5}$$

where $G_{n,s}(f)$ is given by (1.4). This formula is exact for all algebraic polynomials of degree at most $2(s+1)n-1$ (see [13]). The knots τ_v, $v=1,\dots,n$, in (1.5) are zeros of the monic polynomial $\pi_n(\cdot)=\pi_{n,s}(\cdot;w)$, which minimizes the integral $\int_{-1}^{1} \pi_2(t)^{2s+2}w(t)\,\mathrm{d}t$, where $\pi_n(t)=\sum_{k=0}^{n} a_k t^k, a_n=1$. This minimization leads to the "orthogonality conditions"

$$\int_{-1}^{1} t^k \pi_n(t)^{2s+1} w(t)\,\mathrm{d}t = 0, \quad k=0,1,\dots,n-1. \tag{1.6}$$

The polynomials $\pi_{n,s}(\cdot;w)$ are known as *s-orthogonal* (or *s-self associated*) *polynomials* in the interval $[-1,1]$ with respect to the weight function w. For $s=0$ we have the standard case of orthogonal polynomials. For more details of this kind of orthogonality and quadratures with multiple nodes, as well as for a list of references see the survey paper [13].

In the general case, the Gauss–Turán–Lobatto quadrature formula (1.2) has the maximum algebraic degree of precision $2(s+1)n+p+q-1$. Using the nodes in (1.2), we introduce the polynomials

$$\pi_n(t) = \prod_{v=1}^{n} (t-\tau_v) \quad \text{and} \quad u^L(t) = (1+t)^p(1-t)^q.$$

In the cases when $q=0$ or $p=0$ (the Gauss–Turán–Radau case) we use the notation $u^R(t)$ instead of $u^L(t)$. The following characterization is well known: The points $\tau_1,\dots,\tau_n$ are nodes of the Gauss–Turán–Lobatto quadrature formula (1.2) if and only if

$$\int_{-1}^{1} t^k \pi_n(t)^{2s+1} u^L(t) w(t)\,\mathrm{d}t = 0, \quad k=0,1,\dots,n-1. \tag{1.7}$$

These conditions correspond to the "orthogonality conditions" (1.6). A more general result in this direction was given by Stancu [20].

According to (1.7) and (1.6) a construction of Gauss–Turán quadratures can be also applied to the Gauss–Turán–Lobatto and Gauss–Turán–Radau formulas with a little modification of the measure $d\lambda(t) = w(t)\,dt$. Namely, we need a new weight function $w^L(t) = (1+t)^p(1-t)^q w(t)$ in the Gauss–Turán–Lobatto case and $w^R(t) = (1+t)^p w(t)$ or $w^R(t) = (1-t)^q w(t)$ in the Gauss–Turán–Radau case. A connection between the Gauss–Turán–Lobatto formula (1.2) and a Gauss–Turán quadrature, with respect to the weight function $w^L(t)$ on $(-1,1)$, can be given by the following statement (cf. [2,19]):

Lemma 1.1. *For the Gauss–Turán–Lobatto formula* (1.2), *there exists a Gauss–Turán quadrature*

$$\int_{-1}^{1} g(t)w^L(t)\,dt = Q_{n,s}^{\mathrm{T}}(g) + R_{n,s}^{\mathrm{T}}(g), \quad R_{n,s}^{\mathrm{T}}(\mathscr{P}_{2(s+1)n-1}) = 0, \tag{1.8}$$

where

$$w^L(t) = (1+t)^p(1-t)^q w(t), \quad Q_{n,s}^{\mathrm{T}}(g) := \sum_{v=1}^{n}\sum_{i=0}^{2s} A_{i,v}^{\mathrm{T}} g^{(i)}(\tau_v),$$

the nodes τ_v are zeros of s-orthogonal polynomial $\pi_{n,s}(\cdot, w^L)$, while the weights $A_{i,v}^{\mathrm{T}}$ are expressible in terms of those in (1.2) *by*

$$A_{i,v}^{\mathrm{T}} = \sum_{k=i}^{2s} \binom{k}{i} [D^{k-i}((1+t)^p(1-t)^q)]_{t=\tau_v} A_{k,v}, \quad i = 0, 1, \ldots, 2s,$$

where D is the standard differentiation operator.

Let $N = p + q - 1$ and $H_N(t)$ be an interpolating polynomial for f at nodes ∓ 1, with the corresponding multiplicities p and q,

$$H_N(t) = \sum_{i=0}^{p-1} a_i(t) f^{(i)}(-1) + \sum_{i=0}^{q-1} b_i(t) f^{(i)}(1), \quad a_i(t), b_i(t) \in \mathscr{P}_N. \tag{1.9}$$

Then, putting $\phi(t) = f(t) - H_N(t) = u^L(t)g(t)$, we can find the weights α_i and β_i in (1.3). Since $F_{n,s}^{(p,q)}(\phi) = 0$ and $R_{n,s}^{(p,q)}(\phi) = R_{n,s}^{(p,q)}(f)$, we have

$$I(\phi; w) = G_{n,s}(\phi) + R_{n,s}^{(p,q)}(\phi),$$

i.e.,

$$I(f; w) = I(H_N; w) + G_{n,s}(f) - G_{n,s}(H_N) + R_{n,s}^{(p,q)}(f) = Q_{n,s}^{(p,q)}(f) + R_{n,s}^{(p,q)}(f).$$

Thus, $F_{n,s}^{(p,q)}(f) = I(H_N; w) - G_{n,s}(H_N)$, i.e.,

$$\alpha_i = I(a_i; w) - G_{n,s}(a_i), \quad \beta_i = I(b_i; w) - G_{n,s}(b_i).$$

Remark 1.2. Defining the polynomials $P_N^{(v,k)}(x; a, b)$, $0 \leqslant v \leqslant k \leqslant N-1$, by

$$P_N^{(v,k)}(x; a, b) := C_N^{(v,k)}(a, b)(x-a)^v \int_b^x (x-a)^{k-v}(x-b)^{N-k-1}\,dx,$$

where

$$C_N^{(v,k)}(a,b) = \frac{(-1)^{N-k}(N-v)!}{v!(k-v)!(N-k-1)!}(b-a)^{v-N},$$

the polynomials $a_i(t), b_i(t)$, which are appearing in (1.9), can be expressed in the following form

$$a_i(t) = P_N^{(i,p-1)}(t;-1,1), \quad b_i(t) = P_N^{(i,q-1)}(t;1,-1).$$

The connection of the Gauss–Turán–Radau formula with a Gauss–Turán formula can be obtained by putting $p=0$ or $q=0$.

The proof of the existence and the uniqueness of such quadrature rules has been obtained in [19]. Numerically stable procedures for determining the nodes and the coefficients in (1.2) have been given recently in [14,19] (see [13] for earlier references).

In this paper we consider the remainder term $R_{n,s}^{(p,q)}(f)$ of Gauss–Turán–Lobatto and Gauss–Turán–Radau quadrature formulae for classes of analytic functions on elliptical contours in the complex plane containing the interval $[-1,1]$ in its interior. The paper is organized as follows. The remainder term $R_{n,s}^{(p,q)}(f)$ for analytic functions is given in Section 2. The cases of elliptic contours with foci at the points ± 1 for some Jacobi weight functions w are analyzed in Section 3. Asymptotic behavior of the remainder term is also studied. In Section 4 we consider the L^1-type error estimates, including some numerical examples.

2. Remainder term for analytic functions

Let Γ be a simple closed curve in the complex plane surrounding the interval $[-1,1]$ and $\mathscr{D}$ be its interior. Suppose f is an analytic function in $\mathscr{D}$ and continuous on $\bar{\mathscr{D}}$. Taking any system of m distinct points $\{\xi_1,\ldots,\xi_m\}$ in $\mathscr{D}$ and m nonnegative integers $n_1,\ldots,n_m$, the error in the Hermite interpolating polynomial of f at the point t $(\in \mathscr{D})$ can be expressed in the form (see, e.g., Gončarov [9, Chapter 5])

$$r_m(f;t) = f(t) - \sum_{v=1}^{m}\sum_{i=0}^{n_v-1} \ell_{i,v}(t) f^{(i)}(\xi_v) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(z)\Omega_m(t)}{(z-t)\Omega_m(z)}\,\mathrm{d}z,$$

where $\ell_{i,v}(t)$ are the fundamental functions of Hermite interpolation and

$$\Omega_m(z) = \prod_{v=1}^{m}(z-\xi_v)^{n_v}. \tag{2.1}$$

By multiplying this formula with the weight function $w(t)$ and integrating in t over $(-1,1)$ we get a contour integral representation of the remainder term $R_m(f)$ in a quadrature formula with multiple nodes

$$R_m(f) = I(f;w) - \sum_{v=1}^{m}\sum_{i=0}^{n_v-1} A_{i,v} f^{(i)}(\xi_v) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma K_m(z,w) f(z)\,\mathrm{d}z, \tag{2.2}$$

where $A_{i,\nu} = \int_{-1}^{1} \ell_{i,\nu}(t)w(t)\,\mathrm{d}t$ and the kernel $K_m(z,w)$ is given by

$$K_m(z,w) = \frac{\rho_m(z;w)}{\Omega_m(z)}, \quad \rho_m(z;w) = \int_{-1}^{1} \frac{\Omega_m(t)}{z-t}\,w(t)\,\mathrm{d}t, \quad z \in \mathbb{C} \setminus [-1,1]. \tag{2.3}$$

Now, we return to the Gauss–Turán–Lobatto quadrature formula (1.1), so that the node polynomial (which in general depends on w), given by (2.1), in this case ($m = n+2$), becomes

$$\omega_{n+2}(z;w) = (z+1)^p(z-1)^q \prod_{\nu=1}^{n} (z-\tau_\nu)^{2s+1} = (-1)^q u^L(z)[\pi_{n,s}(z;w^L)]^{2s+1}$$

and

$$\rho_{n+2}(z;w) = (-1)^q \int_{-1}^{1} \frac{u^L(t)[\pi_{n,s}(t;w^L)]^{2s+1}}{z-t}\,w(t)\,\mathrm{d}t, \tag{2.4}$$

where $z \in \mathbb{C} \setminus [-1,1]$, $u^L(z) = (1+z)^p(1-z)^q$, $w^L(z) = u^L(z)w(z)$, and $\pi_{n,s}(\cdot;w^L)$ is the s-orthogonal polynomial of degree n with respect to the weight w^L.

The corresponding remainder term (2.2) reduces to

$$R_{n,s}^{(p,q)}(f) = \frac{1}{2\pi \mathrm{i}} \oint_{\Gamma} K_{n,s}^{(p,q)}(z,w) f(z)\,\mathrm{d}z, \tag{2.5}$$

where, according to (2.3) and (2.4),

$$K_{n,s}^{(p,q)}(z,w) = \frac{\rho_{n+2}(z;w)}{\omega_{n+2}(z;w)} = \frac{\rho_n(z;w^L)}{u^L(z)[\pi_{n,s}(z;w^L)]^{2s+1}}. \tag{2.6}$$

Denoting by $K_{n,s}(z,w^L)$ the kernel for the Gauss–Turán quadrature rule (1.8), (2.6) can be written in the following form:

$$K_{n,s}^{(p,q)}(z,w) = \frac{K_{n,s}(z,w^L)}{u^L(z)}. \tag{2.7}$$

The integral representation (2.5) leads to the error estimate

$$|R_{n,s}^{(p,q)}(f)| \leqslant \frac{\ell(\Gamma)}{2\pi} \left(\max_{z \in \Gamma} |K_{n,s}^{(p,q)}(z,w)| \right) \left(\max_{z \in \Gamma} |f(z)| \right), \tag{2.8}$$

where $\ell(\Gamma)$ is the length of the contour Γ. Therefore, in order to get an estimate of the remainder term, it is important to study the magnitude of $|K_{n,s}(z)|$ on the contour Γ. The kernels of the remainder term for Gauss–Turán quadratures for classes of analytic functions on elliptical contours with foci at ± 1 and some special Jacobi weights have been recently studied in [15]. This approach for Gaussian type formulae ($s = 0$) was used by Gautschi and Varga [8] (see also [3–5,7,12,17,18]).

A general estimate of the remainder term can be obtained by applying the Hölder inequality to (2.5). Thus, from

$$|R_{n,s}^{(p,q)}(f)| = \frac{1}{2\pi} \left| \oint_{\Gamma} K_{n,s}^{(p,q)}(z,w) f(z)\,\mathrm{d}z \right| \leqslant \frac{1}{2\pi} \oint_{\Gamma} |K_{n,s}^{(p,q)}(z,w)|\,|f(z)|\,|\mathrm{d}z|$$

for $1 \leqslant r \leqslant +\infty$ and $1/r + 1/r' = 1$, we get

$$|R_{n,s}^{(p,q)}(f)| \leqslant \frac{1}{2\pi}\left(\oint_\Gamma |K_{n,s}^{(p,q)}(z,w)|^r |\mathrm{d}z|\right)^{1/r}\left(\oint_\Gamma |f(z)|^{r'}|\mathrm{d}z|\right)^{1/r'},$$

i.e.,

$$|R_{n,s}^{(p,q)}(f)| \leqslant \frac{1}{2\pi}\|K_{n,s}^{(p,q)}(\cdot,w)\|_r \|f\|_{r'},$$

where we put

$$\|f\|_r := \begin{cases} \left(\oint_\Gamma |f(z)|^r|\mathrm{d}z|\right)^{1/r}, & 1 \leqslant r < +\infty, \\ \max_{z\in\Gamma} |f(z)|, & r = +\infty. \end{cases}$$

Taking $r = +\infty$, the above estimate becomes

$$|R_{n,s}^{(p,q)}(f)| \leqslant \frac{1}{2\pi}\|K_{n,s}^{(p,q)}(\cdot,w)\|_\infty \|f\|_1 \leqslant \frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}^{(p,q)}(z,w)|\right)\|f\|_\infty,$$

i.e., (2.8).

On the other side, for $r = 1$ we have

$$|R_{n,s}^{(p,q)}(f)| \leqslant \frac{1}{2\pi}\|K_{n,s}^{(p,q)}(\cdot,w)\|_1 \|f\|_\infty,$$

i.e.,

$$|R_{n,s}^{(p,q)}(f)| \leqslant \frac{1}{2\pi}\left(\oint_\Gamma |K_{n,s}^{(p,q)}(z,w)|\,|\mathrm{d}z|\right)\|f\|_\infty. \tag{2.9}$$

This L^1-type of estimates for the Gaussian quadratures on elliptical contours for the Chebyshev weights was investigated by Hunter [11].

Also, the L^2-estimate,

$$|R_{n,s}^{(p,q)}(f)| \leqslant \frac{1}{2\pi}\|K_{n,s}^{(p,q})(\cdot,w)\|_2 \|f\|_2,$$

can be of some interest. Such estimates will be given elsewhere.

3. Estimates of the remainder term on confocal ellipses for a subclass of Jacobi weights and an asymptotic behavior

In this section we take as the contour Γ an ellipse with foci at the points ± 1 and sum of semiaxes $\varrho > 1$,

$$\mathscr{E}_\varrho = \{z \in \mathbb{C} : z = \tfrac{1}{2}(\varrho \mathrm{e}^{\mathrm{i}\theta} + \varrho^{-1}\mathrm{e}^{-\mathrm{i}\theta}),\ 0 \leqslant \theta < 2\pi\}.$$

When $\varrho \to 1$, then the ellipse shrinks to the interval $[-1, 1]$, while with increasing ϱ it becomes more and more circle-like. There is also another choice of the contour Γ as a circle $\mathscr{C}_\varrho$ with center at origin and radius ϱ (> 1). The advantage of the elliptical contours is that such choice needs the analyticity of f in a smaller region of the complex plane, especially when ϱ is near to 1.

Since the ellipse $\mathscr{E}_\varrho$ has length $\ell(\mathscr{E}_\varrho) = 4\varepsilon^{-1}E(\varepsilon)$, where ε is the eccentricity of $\mathscr{E}_\varrho$, i.e., $\varepsilon = 2/(\varrho + \varrho^{-1})$, and

$$E(\varepsilon) = \int_0^{\pi/2} \sqrt{1 - \varepsilon^2 \sin^2\theta}\,\mathrm{d}\theta$$

is the complete elliptic integral of the second kind, estimate (2.8) reduces to

$$|R_{n,s}(f)| \leqslant \frac{2E(\varepsilon)}{\pi\varepsilon}\left(\max_{z\in\mathscr{E}_\varrho} |K_{n,s}(z)|\right)\|f\|_\infty, \quad \varepsilon = \frac{2}{\varrho + \varrho^{-1}}. \tag{3.1}$$

Note that the bound on the right in (3.1) is a function of ϱ, so that it can be optimized with respect to $\varrho > 1$.

It is well known that the (monic) Chebyshev polynomials of the first kind T_n, orthogonal with respect to $w_1(t) = (1-t^2)^{-1/2}$ on $(-1,1)$, are also s-orthogonal relative to the same weight on $(-1,1)$ for each $s \geqslant 0$ (Bernstein [1]). In 1975 Ossicini and Rosati [16] showed that for three other Jacobi weights (but depending on s),

$$w_2(t) = (1-t^2)^{1/2+s}, \quad w_3(t) = \frac{(1+t)^{1/2+s}}{(1-t)^{1/2}}, \quad w_4(t) = \frac{(1-t)^{1/2+s}}{(1+t)^{1/2}},$$

Chebyshev polynomials of the second kind U_n, the third kind V_n, and the fourth kind W_n appeared as s-orthogonal, respectively. These polynomials are defined by

$$U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}, \quad V_n(\cos\theta) = \frac{\cos(n+\frac{1}{2})\theta}{\cos\frac{1}{2}\theta}, \quad W_n(\cos\theta) = \frac{\sin(n+\frac{1}{2})\theta}{\sin\frac{1}{2}\theta}$$

(cf. Gautschi and Notaris [6]). It is easy to see that $W_n(-t) = (-1)^n V_n(t)$, so that the last weight w_4 can be omitted from our investigation.

In our analysis of the remainder term $R_{n,s}^{(p,q)}(f)$ in the Gauss–Turán–Lobatto quadrature formulae (1.1), we consider the following subclass of Jacobi weights:

$$\mathscr{F}_s := \{w(t) = (1-t)^{k-1/2}(1+t)^{m-1/2} : m,k \in \mathbb{N}_0,\ 0 \leqslant m,k \leqslant s+1\}.$$

Regarding the previous result of Ossicini and Rosati [16], our aim is to identify the weights from $\mathscr{F}_s$ for which the s-orthogonal polynomials are T_n, U_n, and V_n.

Since $u^L(z) = (1+z)^p(1-z)^q$ $(p,q \geqslant 0)$, according to results from Section 2 we have, for each $w \in \mathscr{F}_s$,

$$w^L(z) = (1-z)^{k+q-1/2}(1+z)^{m+p-1/2}, \quad 0 \leqslant k,m \leqslant s+1.$$

Since $z \in \mathscr{E}_\varrho$, we put $u = \varrho e^{i\theta}$, so that $z = \frac{1}{2}(u + u^{-1})$. In sequel we need the following notation:

$$a_\nu = a_\nu(\varrho) = \tfrac{1}{2}(\varrho^\nu + \varrho^{-\nu}), \quad \nu \in \mathbb{N},\ \varrho > 1. \tag{3.2}$$

Note that $a_\nu > 1$ and $a_{2\nu} = 2a_\nu^2 - 1$, as well as

$$|u^\nu - u^{-\nu}|^2 = 2(a_{2\nu} - \cos 2\nu\theta), \quad |u^\nu + u^{-\nu}|^2 = 2(a_{2\nu} + \cos 2\nu\theta).$$

3.1. Case $\pi_{n,s}(t) = 2^{-n+1}T_n(t)$

This is a trivial case, because only one weight produces such s-orthogonal polynomials ($p=q=k=m=0$). It is a well known Chebyshev weight $w_1(t)=(1-t^2)^{-1/2}$ and it enables only the Gauss–Turán quadratures ($p=q=0$). This case has recently been investigated in [15] and a conjecture was stated that *for each fixed $\varrho > 1$ and $s \in \mathbb{N}_0$ there exists $n_0 = n_0(\varrho, s) \in \mathbb{N}$ such that*

$$\max_{z\in\mathscr{E}_\varrho} |K_{n,s}(z)| = K_{n,s}(\tfrac{1}{2}(\varrho+\varrho^{-1}))$$

for each $n \geqslant n_0$.

3.2. Case $\pi_{n,s}(t) = 2^{-n}U_n(t)$

From $k+q-\frac{1}{2} = m+p-\frac{1}{2} = \frac{1}{2}+s$, we get

$$p = s+1-m, \quad q = s+1-k \quad (0 \leqslant k,m \leqslant s+1). \tag{3.3}$$

Thus, for each $w \in \mathscr{F}_s$ we can analyze the remainder term $R_{n,s}^{(p,q)}(f)$ in the Gauss–Turán–Lobatto quadrature rule

$$\int_{-1}^{1} f(t)(1-t)^{k-1/2}(1+t)^{m-1/2}\mathrm{d}t = Q_{n,s}^{(p,q)}(f) + R_{n,s}^{(p,q)}(f), \tag{3.4}$$

where the multiplicities p and q of the fixed nodes ∓ 1 are given by (3.3) and

$$Q_{n,s}^{(p,q)}(f) = \sum_{i=0}^{s-m} \alpha_i f^{(i)}(-1) + \sum_{i=0}^{s-k} \beta_i f^{(i)}(1) + \sum_{v=1}^{n}\sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v). \tag{3.5}$$

Remark 3.1. For $k=m=0$, (3.4) reduces to a symmetric Gauss–Turán–Lobatto quadrature formula with the Chebyshev weight of the first kind. Putting $m=s+1$ or $k=s+1$, one of sums with fixed nodes in (3.5) vanish, so that formula (3.4) becomes a Gauss–Turán–Radau type formula.

According to (2.7), the kernel for the quadrature rule (3.4) becomes

$$K_{n,s}^{(p,q)}(z,w) = \frac{K_{n,s}(z,w_2)}{u^L(z)} = \frac{(1-z)^k(1+z)^m}{(1-z^2)^{s+1}} K_{n,s}(z,w_2).$$

Since $z \in \mathscr{E}_\varrho$ and $u = \varrho e^{i\theta}$, using notation (3.2) we have

$$|1-z| = a_1 - \cos\theta, \quad |1+z| = a_1 + \cos\theta, \quad |1-z^2| = \tfrac{1}{2}(a_2 - \cos 2\theta)$$

and then

$$|K_{n,s}^{(p,q)}(z,w)| = \frac{2^{s+1}(a_1-\cos\theta)^k(a_1+\cos\theta)^m}{(a_2-\cos 2\theta)^{s+1}} |K_{n,s}(z,w_2)|. \tag{3.6}$$

An analysis in details of $|K_{n,s}^{(p,q)}(z,w)|$, when $z \in \mathscr{E}_\varrho$, can be given as in [15]. In order to prove the convergence of the Gauss–Turán–Lobatto quadrature rule (3.5), when $n \to +\infty$ or $s \to +\infty$, a crude

Table 1

n	1	2	3	4	5	6	7	8	9	10
ϱ_n	1.932	1.654	1.515	1.430	1.372	1.330	1.297	1.271	1.249	1.231
n	11	12	13	14	15	16	17	18	19	20
ϱ_n	1.216	1.203	1.191	1.181	1.172	1.164	1.157	1.151	1.145	1.139

estimate of $|K_{n,s}(z,w_2)|$ from [15] can be used. Namely,

$$|K_{n,s}(z,w_2)| \leqslant \frac{\pi}{\varrho^{n+1}}\left(\frac{\varrho+\varrho^{-1}}{\varrho^{n+1}-\varrho^{-(n+1)}}\right)^{2s+1} = \frac{\pi}{\varrho^{n+1}}\left(\frac{a_2+1}{a_{2n+2}-1}\right)^{s+1/2}.$$

In this way, we obtain and

$$|K_{n,s}^{(p,q)}(z,w)| \leqslant \frac{(a_1+1)^{k+m}\pi}{\varrho^{n+1}}\left(\frac{2}{a_2-1}\right)^{s+1}\left(\frac{a_2+1}{a_{2n+2}-1}\right)^{s+1/2}$$

and

$$|R_{n,s}^{(p,q)}(f)| \leqslant \frac{M}{\varrho^{n+1}}\left(\frac{2}{a_2-1}\right)^{s+1}\left(\frac{a_2+1}{a_{2n+2}-1}\right)^{s+1/2}, \tag{3.7}$$

where M is a constant ($M = 2a_1(a_1+1)^{k+m}E(\varepsilon)\|f\|_\infty$).

According to (3.7) we may conclude that the corresponding quadrature formulae converge if s is a fixed integer and $n \to +\infty$. Moreover, $R_{n,s}^{(p,q)}(f) = \mathrm{O}(\varrho^{-2n(s+1)})$, when $n \to +\infty$.

In order to have

$$\lim_{s\to+\infty} R_{n,s}^{(p,q)}(f) = 0$$

when n is fixed, from (3.7) we conclude that it is sufficient to be satisfied the condition

$$\frac{2}{a_2-1}\cdot\frac{a_2+1}{a_{2n+2}-1} < 1, \tag{3.8}$$

i.e., $2(\varrho+\varrho^{-1}) < (\varrho-\varrho^{-1})(\varrho^{n+1}+\varrho^{-n-1})$, because of $a_{2\nu} = 2a_\nu^2 - 1$ and $a_\nu^2 - 1 = (\varrho^\nu - \varrho^{-\nu})^2/4$. It is equivalent to

$$\varrho^{2n+4} > h(\varrho) = \varrho^{2n+2} + 2\varrho^{n+3} + 2\varrho^{n+1} + \varrho^2 - 1.$$

It is easy to conclude that for each $n \in \mathbb{N}$ there exists a unique zero ϱ_n of the equation $\varrho^{2n+4} = h(\varrho)$ in $(1, +\infty)$, so that the sufficient condition (3.8) for the convergence of the Gauss–Turán–Lobatto rules (3.5), when n is fixed and $s \to +\infty$, can be presented in the following form:

$$\varrho > \varrho_n.$$

Numerical values of ϱ_n for $1 \leqslant n \leqslant 20$ are presented in Table 1. Evidently, ϱ_n is an decreasing sequence, converging to 1. For example, $\varrho_{100} = 1.03976$ and $\varrho_{1000} = 1.00585$.

3.3. Case $\pi_{n,s}(t) = 2^{-n} V_n(t)$

From $k+q-\frac{1}{2} = -\frac{1}{2}$ and $m+p-\frac{1}{2} = \frac{1}{2}+s$, we find

$$p = s+1-m, \quad q = 0 \quad (k=0,\ 0 \leqslant m \leqslant s+1).$$

In this case, for $w(t) = (1-t)^{-1/2}(1+t)^{m-1/2}$ $(\in \mathscr{F}_s)$ we analyze the remainder term $R_{n,s}^{(p,q)}(f)$ in the Gauss–Turán–Radau quadrature rule

$$\int_{-1}^{1} f(t)\,\frac{(1-t)^{m-1/2}}{(1-t)^{1/2}}\mathrm{d}t = \sum_{i=0}^{s-m} \alpha_i f^{(i)}(-1) + \sum_{v=1}^{n}\sum_{i=0}^{2s} A_{i,v} f^{(i)}(\tau_v) + R_{n,s}^{(p,q)}(f). \tag{3.9}$$

For $m = 0$ this formula reduces to one relative to the Chebyshev weight of the first kind. According to (2.7), the kernel for the quadrature rule (3.9) becomes

$$K_{n,s}^{(s+1-m,0)}(z,w) = \frac{K_{n,s}(z,w_3)}{(1+z)^{s+1-m}}.$$

Using the estimate (see [15])

$$|K_{n,s}(z;w_3)| \leqslant \frac{2^{s+1}\pi}{\varrho^{n+1/2}} \cdot \frac{a_1+1}{\sqrt{(a_2-1)(a_{2n+1}+1)}}\left(\frac{a_1+1}{a_{2n+1}-1}\right)^s,$$

we obtain

$$|R_{n,s}^{(s+1-m,0)}(f)| \leqslant \frac{M}{\varrho^{n+1/2}\sqrt{a_{2n+1}+1}}\left(\frac{2(a_1+1)}{(a_{2n+1}-1)(a_1-1)}\right)^s, \tag{3.10}$$

where M is a constant $(M = 4a_1(a_1+1)(a_1-1)^{m-1}E(\varepsilon)\|f\|_\infty/\sqrt{a_2-1})$. It is easy to show that $\lim_{n\to+\infty} R_{n,s}^{(s+1-m,0)}(f) = 0$, when s is a fixed integer. Moreover, $R_{n,s}^{(s+1-m,0)}(f) = \mathrm{O}(\varrho^{-2n(s+1)})$, $n \to +\infty$.

A sufficient condition for the convergence of quadrature formulae of Gauss–Turán–Radau type (3.9) in s, when n is a fixed number, can be derived from (3.10) in the form

$$\frac{2(a_1+1)}{(a_{2n+1}-1)(a_1-1)} < 1, \tag{3.11}$$

which is equivalent to the condition

$$\varrho^{2n+2} > h^*(\varrho) = \varrho^{2n+1} + 2\varrho^{n+3/2} + 2\varrho^{n+1/2} + \varrho - 1.$$

As in the previous case, we conclude that for each $n \in \mathbb{N}$ there exists a unique zero ϱ_n^* of the equation $\varrho^{2n+2} = h^*(\varrho)$ in $(1, +\infty)$, so that the sufficient condition (3.11) for the convergence of the Gauss–Turán–Radau rules (3.9), when n is fixed and $s \to +\infty$, can take the form

$$\varrho > \varrho_n^*.$$

Numerical values of ϱ_n^* for $1 \leqslant n \leqslant 20$ are presented in Table 2. Here, also ϱ_n^* is a decreasing sequence, converging to 1. For example, $\varrho_{100}^* = 1.04573$.

Table 2

n	1	2	3	4	5	6	7	8	9	10
ϱ_n^*	2.736	2.046	1.768	1.614	1.516	1.447	1.396	1.356	1.324	1.298
n	11	12	13	14	15	16	17	18	19	20
ϱ_n^*	1.276	1.258	1.242	1.228	1.216	1.205	1.195	1.186	1.178	1.171

4. L^1-type error estimates

Now, we return to the error estimates considered in Section 2. First, by comparing two esitmates, obtained for $r=+\infty$ and 1, i.e., inequalities (2.8) and (2.9), respectively, we can conclude that the second one is stronger inequality. Obviously,

$$L_{n,s}^{(p,q)}(\Gamma) := \frac{1}{2\pi}\oint_\Gamma |K_{n,s}^{(p,q)}(z,w)||\mathrm{d}z| \leqslant \frac{\ell(\Gamma)}{2\pi}\left(\max_{z\in\Gamma}|K_{n,s}^{(p,q)}(z,w)|\right), \tag{4.1}$$

so that this L^1 approach gives better estimates of the remainder term than the alternative approach over (2.8). Therefore, in this section we consider such kind of estimates. Again, we take confocal ellipses $\mathscr{E}_\varrho$ and put $z=\frac{1}{2}(u+u^{-1})$, where $u=\varrho e^{i\theta}$. Then, the left-hand side in (4.1) becomes

$$L_{n,s}^{(p,q)}(\mathscr{E}_\varrho) = \frac{1}{2\pi\sqrt{2}}\int_0^{2\pi} |K_{n,s}^{(p,q)}(z,w)|\sqrt{a_2-\cos 2\theta}\,\mathrm{d}\theta, \tag{4.2}$$

where a_v is defined in (3.2).

We concentrate in this section on bounds based on (4.2). This integral can be evaluated numerically by using a quadrature formula. However, if $w(t)$ are some of generalized Chebyshev weight functions as in Section 3 we can obtain explicit expressions for $L_{n,s}^{(p,q)}(\mathscr{E}_\varrho)$ or for their bounds.

At first, we consider (3.4) in the simplest case when $k=m=s=0$, i.e.,

$$\int_{-1}^{1}\frac{1}{\sqrt{1-t^2}}f(t)\,\mathrm{d}t = \alpha_0 f(-1)+\beta_0 f(1)+\sum_{v=1}^{n}A_{0,v}f(\tau_v)+R_{n,0}^{(1,1)}(f)$$

for which the remainder term on the confocal ellipses has recently been analyzed by Gautschi (see [3]).

Since $|K_{n,s}^{(p,q)}(z,w)|$ is given by (3.6) and

$$|K_{n,0}(z,w_2)| = \frac{\pi}{\varrho^{n+1}}\left(\frac{a_2-\cos 2\theta}{a_{2n+2}-\cos(2n+2)\theta}\right)^{1/2}$$

(cf. [15]), from the general expression (4.2) we obtain

$$L_{n,0}^{(1,1)}(\mathscr{E}_\varrho) = \frac{1}{2\pi\sqrt{2}}\frac{2\pi}{\varrho^{n+1}}\int_0^{2\pi}\frac{\mathrm{d}\theta}{\sqrt{a_{2n+2}-\cos(2n+2)\theta}},$$

i.e.,

$$L_{n,0}^{(1,1)}(\mathscr{E}_\varrho) = \frac{1}{2\varrho^{n+1}}\int_0^{2\pi}\frac{\mathrm{d}\theta}{\sqrt{a_{n+1}^2-\cos^2(n+1)\theta}} = \frac{1}{2(n+1)\varrho^{n+1}}\int_0^{2(n+1)\pi}\frac{\mathrm{d}\theta}{\sqrt{a_{n+1}^2-\cos^2\theta}},$$

because of $a_{2n+2}=2a_{n+1}^2-1$. Since the integrand is a periodic and an even function we obtain

$$L_{n,0}^{(1,1)}(\mathscr{E}_\varrho)=\frac{1}{\varrho^{n+1}}\int_0^{\pi}\frac{\mathrm{d}\theta}{\sqrt{a_{n+1}^2-\cos^2\theta}}=\frac{2}{\varrho^{n+1}}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{a_{n+1}^2-\cos^2\theta}}.$$

Therefore,

$$L_{n,0}^{(1,1)}(\mathscr{E}_\varrho)=\frac{2}{\varrho^{n+1}}\int_0^{\pi/2}\frac{\mathrm{d}\theta}{\sqrt{a_{n+1}^2-\sin^2\theta}}=\frac{2}{a_{n+1}\varrho^{n+1}}K\left(\frac{1}{a_{n+1}}\right),$$

where K is the complete elliptic integral of the first kind, i.e.,

$$K(k)=\int_0^{\pi/2}(1-k^2\sin^2\theta)^{-1/2}\,\mathrm{d}\theta\quad(|k|<1).$$

Finally, by substituting $a_{n+1}=(\varrho^{n+1}+\varrho^{-n-1})/2$, we get

$$L_{n,0}^{(1,1)}(\mathscr{E}_\varrho)=\frac{4}{\varrho^{2n+2}+1}K\left(\frac{2}{\varrho^{n+1}+\varrho^{-n-1}}\right). \tag{4.3}$$

Putting $m=0,k=1$, and $s=0$, (3.4) reduces to a formula of Radau type, i.e.,

$$\int_{-1}^{1}\frac{(1-t)^{1/2}}{(1+t)^{-1/2}}f(t)\,\mathrm{d}t=\alpha_0 f(-1)+\sum_{v=1}^{n}A_{0,v}f(\tau_v)+R_{n,0}^{(1,0)}(f).$$

Then,

$$|K_{n,0}^{(1,0)}(z,w)|=\frac{2(a_1-\cos\theta)}{a_2-\cos 2\theta}|K_{n,0}(z,w_2)|$$

and we get

$$L_{n,0}^{(1,0)}(\mathscr{E}_\varrho)=\frac{1}{2\varrho^{n+1}}\int_0^{2\pi}\frac{a_1-\cos\theta}{\sqrt{a_{n+1}^2-\cos^2(n+1)\theta}}\,\mathrm{d}\theta.$$

Let $v(\theta):=(a_{n+1}^2-\cos^2(n+1)\theta)^{-1/2}\cos\theta$. Since $v(\theta+\pi)=-v(\theta)$ and

$$\int_0^{2\pi}\frac{\cos\theta}{\sqrt{a_{n+1}^2-\cos^2(n+1)\theta}}\mathrm{d}\theta=\int_0^{\pi}[v(\theta)+v(\theta+\pi)]\,\mathrm{d}\theta=0,$$

we obtain

$$L_{n,0}^{(1,0)}(\mathscr{E}_\varrho)=\frac{a_1}{2\varrho^{n+1}}\int_0^{2\pi}\frac{\mathrm{d}\theta}{\sqrt{a_{n+1}^2-\cos^2(n+1)\theta}}=a_1L_{n,0}^{(1,1)}(\mathscr{E}_\varrho),$$

where $L_{n,0}^{(1,1)}(\mathscr{E}_\varrho)$ is given in (4.3). Therefore,

$$L_{n,0}^{(1,0)}(\mathscr{E}_\varrho)=\frac{2(\varrho+\varrho^{-1})}{\varrho^{2n+2}+1}K\left(\frac{2}{\varrho^{n+1}+\varrho^{-n-1}}\right). \tag{4.4}$$

In a similar way, we can analyze the Gauss–Radau type formula from (3.9). Putting $m=s=0$, (3.9) reduces to

$$\int_{-1}^{1} \frac{1}{\sqrt{1-t^2}} f(t)\,\mathrm{d}t = \alpha_0 f(-1) + \sum_{v=1}^{n} A_{0,v} f(\tau_v) + R_{n,0}^{(1,0)}(f)$$

with the kernel $K_{n,0}^{(1,0)}(z,w) = K_{n,0}(z,w_3)/(1+z)$. Using

$$K_{n,0}(z,w_3) = \frac{2\pi}{\varrho^{n+1/2}} \cdot \frac{a_1 + \cos\theta}{(a_2 - \cos 2\theta)^{1/2}(a_{2n+1} + \cos(2n+1)\theta)^{1/2}}$$

(see [15, Eq. (3.20)] for $s=0$) and (4.2), we obtain

$$L_{n,0}^{(1,0)}(\mathscr{E}_\varrho) = \frac{1}{\sqrt{2}\varrho^{n+1/2}} \int_0^{2\pi} \frac{\mathrm{d}\theta}{\sqrt{a_{2n+1} + \cos(2n+1)\theta}}.$$

Since the integrand is an even and periodic function, it reduces to

$$L_{n,0}^{(1,0)}(\mathscr{E}_\varrho) = \frac{\sqrt{2}}{\varrho^{n+1/2}} \int_0^{\pi} \frac{\mathrm{d}\theta}{\sqrt{a_{2n+1} + \cos\theta}}.$$

Finally, by substitution $a_{2n+1} = \frac{1}{2}(\varrho^{2n+1} + \varrho^{-2n-1}) = 2q_n^2 - 1$, where $q_n = \frac{1}{2}(\varrho^{n+1/2} + \varrho^{-n-1/2})$, this integral becomes

$$L_{n,0}^{(1,0)}(\mathscr{E}_\varrho) = \frac{1}{\varrho^{n+1/2}} \int_0^{\pi} \frac{\mathrm{d}\theta}{\sqrt{q_n^2 - \sin^2\frac{\theta}{2}}} = \frac{2}{\varrho^{n+1/2} q_n} K\left(\frac{1}{q_n}\right),$$

i.e.,

$$L_{n,0}^{(1,0)}(\mathscr{E}_\varrho) = \frac{4}{\varrho^{2n+1}+1} K\left(\frac{2}{\varrho^{n+1/2} + \varrho^{-n-1/2}}\right). \tag{4.5}$$

Thus, in the case of Gaussian type quadratures ($s=0$), formulae (4.3)–(4.5) give the exact values of the corresponding quantities $L_{n,0}^{(p,q)}$ in terms of the complete elliptic integral of the first kind.

In order to consider a general case of (3.4) we need two following integrals:

$$J_k(a) = \int_0^{\pi} \frac{\cos k\theta}{(a+\cos\theta)^{2s+1}}\,\mathrm{d}\theta, \quad I_{k,m}(a) = \int_0^{\pi} (a-\cos\theta)^{2k}(a+\cos\theta)^{2m}\,\mathrm{d}\theta,$$

where $a > 1$ and $s, m, k \geqslant 0$.

Lemma 4.1. *Let* $x > 1$, $a = (x+1)/(2\sqrt{x})$, *and* $s \in \mathbb{N}_0$. *Then*

$$J_k(a) = \frac{2^{2s+1}\pi(-1)^k x^{s-(k-1)/2}}{(x-1)^{4s+1}} \sum_{v=0}^{2s} \binom{2s+v}{v} \binom{2s+k}{k+v} (x-1)^{2s-v}. \tag{4.6}$$

This result can be found in the book [10, Eq. 3.616.7].

Lemma 4.2. *Let*

$$A_0^{(2k)} = \sum_{j=0}^{k} \frac{1}{4^j} \binom{2k}{2j} \binom{2j}{j} a^{2k-2j},$$

$$A_\nu^{(2k)} = \frac{(-1)^\nu}{2^{\nu-1}} \sum_{j=0}^{[(2k-\nu)/2]} \frac{1}{4^j} \binom{2k}{\nu+2j} \binom{\nu+2j}{j} a^{2k-\nu-2j}, \quad \nu = 1, \ldots, 2k.$$

Then $I_{k,m} = \pi C_{k,m}(a)$ $(m, k \geqslant 0)$, *where*

$$C_{k,m} = C_{k,m}(a) = A_0^{(2k)} A_0^{(2m)} + \frac{1}{2} \sum_{\nu=1}^{2\min\{k,m\}} (-1)^\nu A_\nu^{(2k)} A_\nu^{(2m)}.$$

Proof. Let $A_\nu^{(2k)}$ be coefficients in the following expansion in the Chebyshev polynomials

$$(a-x)^{2k} = \sum_{\nu=0}^{2k} A_\nu^{(2k)} T_\nu(x), \quad -1 \leqslant x \leqslant 1.$$

Then,

$$I_{k,m} = \int_{-1}^{1} \frac{(a-x)^{2k}(a+x)^{2m}}{\sqrt{1-x^2}}\,\mathrm{d}x$$

and after some tedious calculations, the integral can take the desired form. □

Notice that $C_{k,m} = C_{m,k}$. We list here $C_{k,m}$ for some values of k, m $(k \leqslant m)$:

$$C_{0,0} = 1, \quad C_{0,1} = \tfrac{1}{2}(2a^2+1), \quad C_{0,2} = \tfrac{1}{8}(8a^4 + 24a^2 + 3),$$

$$C_{1,1} = \tfrac{1}{8}(8a^4 - 8a^2 + 3), \quad C_{1,2} = \tfrac{1}{16}(16a^6 - 8a^4 - 6a^2 + 5),$$

$$C_{2,2} = \tfrac{1}{128}(128a^8 - 256a^6 + 288a^4 - 160a^2 + 35).$$

Now, we are ready to estimate quantity (4.2) for the Gauss–Turán–Lobatto quadrature rule (3.4), with (3.5). According to (4.2) and (3.6) we have

$$L_{n,s}^{(p,q)}(\mathscr{E}_\varrho) = \frac{2^s}{\pi\sqrt{2}} \int_0^{2\pi} \frac{(a_1 - \cos\theta)^k (a_1 + \cos\theta)^m}{(a_2 - \cos 2\theta)^{s+1/2}} |K_{n,s}(z, w_2)|\,\mathrm{d}\theta, \tag{4.7}$$

where (see [15])

$$|K_{n,s}(z, w_2)| = \frac{\pi}{4^s \varrho^{n+1}} \left(\frac{a_2 - \cos 2\theta}{a_{2n+2} - \cos(2n+2)\theta}\right)^{s+1/2} |Z_{n,s}^{(2)}(\varrho \mathrm{e}^{\mathrm{i}\theta})|$$

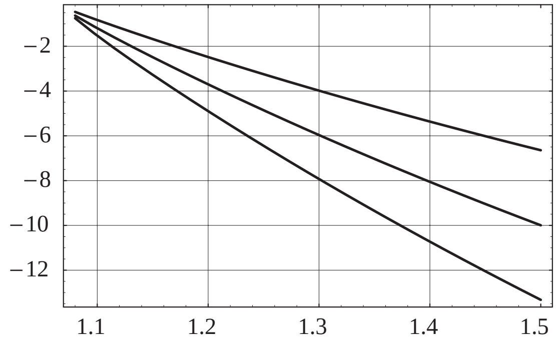

Fig. 1. Log_{10} of the values $L_{10,s}^{(s,s)}(\mathscr{E}_\varrho)$ as functions of ϱ, for $s = 1, 2, 3$.

and

$$Z_{n,s}^{(2)}(u) = \sum_{v=0}^{s} (-1)^{s-v} \binom{2s+1}{v} u^{2(n+1)(v-s)}, \quad u = \varrho e^{i\theta}.$$

Since

$$\begin{aligned} |Z_{n,s}^{(2)}(\varrho e^{i\theta})|^2 &= Z_{n,s}^{(2)}(\varrho e^{i\theta}) Z_{n,s}^{(2)}(\varrho e^{-i\theta}) \\ &= \frac{1}{\varrho^{2(n+1)s}} \sum_{v,\mu=0}^{s} (-1)^{v+\mu} \binom{2s+1}{v} \binom{2s+1}{\mu} \varrho^{2(n+1)(v+\mu-s)} e^{2i(v-\mu)(n+1)\theta} \\ &= \frac{1}{\varrho^{2(n+1)s}} \sum_{\ell=0}^{s} A_\ell \cos 2\ell(n+1)\theta, \end{aligned}$$

where

$$A_\ell = \sum_{\substack{|v-\mu|=\ell \\ v,\mu=0,1,\ldots,s}} (-1)^{v+\mu} \binom{2s+1}{v} \binom{2s+1}{\mu} \varrho^{2(n+1)(v+\mu-s)}, \quad \ell = 0, 1, \ldots, s, \tag{4.8}$$

(4.7) reduces to

$$L_{n,s}^{(p,q)}(\mathscr{E}_\varrho) = \frac{1}{2^{s-1/2} \varrho^{(n+1)(s+1)}} \int_0^\pi \frac{\left(\sum_{\ell=0}^{s} A_\ell \cos 2\ell(n+1)\theta \right)^{1/2}}{(a_{2n+2} - \cos 2(n+1)\theta)^{s+1/2}} \gamma(\theta)\, d\theta, \tag{4.9}$$

where $\gamma(\theta) := (a_1 - \cos\theta)^k (a_1 + \cos\theta)^m$.

As an example, the graphs of functions $\varrho \mapsto \log_{10}(L_{n,s}^{(p,q)}(\mathscr{E}_\varrho))$, for $s = 1, 2, 3$, when $n = 10$, $k = m = 1$ ($p = q = s$), are displayed in Fig. 1. The upper graphs correspond to the smaller values of s.

The following theorem gives a bound of $L_{n,s}^{(p,q)}(\mathscr{E}_\varrho)$.

Theorem 4.3. *Let a_j and A_ℓ be defined by* (3.2) *and* (4.8), *respectively. Then, for the Gauss–Turán–Lobatto quadrature formula* (3.4), *with* (3.5), *where the multiplicities p and q of the fixed nodes ∓ 1 are given by* (3.3), *we have an estimate of $L_{n,s}^{(p,q)}(\mathscr{E}_\varrho)$ in the form*

$$L_{n,s}^{(p,q)}(\mathscr{E}_\varrho) \leqslant 2\pi\sqrt{C_{k,m}(a_1)}\,\Phi_s\left(\varrho^{4(n+1)}\right), \quad \Phi_s(x) = \sqrt{\frac{P_s(x)}{(x-1)^{4s+1}}}, \tag{4.10}$$

where $P_s(x)$ is an algebraic polynomial of degree $3s$.

Proof. Here, we use Lemmas 4.1 and 4.2. Since $a_{2n+2} = (\varrho^{2n+2} + \varrho^{-2n-2})/2 = (x+1)/(2\sqrt{x})$, we have $x = \varrho^{4(n+1)}$. Also, we can conclude that the coefficients A_ℓ can be expressed in the form

$$A_0 = \frac{1}{x^{s/2}}\sum_{v=0}^{s}\binom{2s+1}{v}^2 x^v, \quad A_\ell = \frac{2(-1)^\ell}{x^{(s-\ell)/2}}\sum_{v=0}^{s-\ell}\binom{2s+1}{v}\binom{2s+1}{v+\ell}x^v, \quad \ell \geqslant 1.$$

Applying the Cauchy inequality to (4.9) we obtain

$$L_{n,s}^{(p,q)}(\mathscr{E}_\varrho) \leqslant \frac{H(x)^{1/2}}{2^{s-1/2}\varrho^{(n+1)(s+1)}}\left(\int_0^\pi \gamma(\theta)^2\,\mathrm{d}\theta\right)^{1/2} = \frac{\sqrt{\pi C_{k,m}(a_1)H(x)}}{2^{s-1/2}\varrho^{(n+1)(s+1)}}, \tag{4.11}$$

where $C_{k,m}(a)$ is given in Lemma 4.2 and

$$H(x) = \int_0^\pi \frac{\sum_{\ell=0}^{s} A_\ell \cos 2\ell(n+1)\theta}{(a_{2n+2} - \cos 2(n+1)\theta)^{2s+1}}\,\mathrm{d}\theta.$$

Because of a periodicity of the integrand in $H(x)$, it reduces to

$$H(x) = \int_0^\pi \frac{\sum_{\ell=0}^{s} A_\ell \cos \ell\theta}{(a_{2n+2} - \cos\theta)^{2s+1}}\,\mathrm{d}\theta.$$

i.e.,

$$H(x) = \sum_{\ell=0}^{s}(-1)^\ell A_\ell \int_0^\pi \frac{\cos\ell\theta}{(a_{2n+2}+\cos\theta)^{2s+1}}\,\mathrm{d}\theta = \sum_{\ell=0}^{s}(-1)^\ell A_\ell J_\ell(a_{2n+2}),$$

where the integrals $J_\ell(a_{2n+2})$ can be expressed in form (4.6). In this way, we obtain

$$H(x) = \sum_{\ell=0}^{s}(-1)^\ell A_\ell \frac{2^{2s+1}\pi(-1)^\ell x^{s-(\ell-1)/2}}{(x-1)^{4s+1}}\sum_{v=0}^{2s}\binom{2s+v}{v}\binom{2s+\ell}{\ell+v}(x-1)^{2s-v}.$$

Now, a substitution of the values of coefficients A_ℓ, $\ell = 0,1,\ldots,s$, leads to

$$H(x) = \frac{2^{2s+1}\pi}{(x-1)^{4s+1}}\sum_{\ell=0}^{s}{}' \frac{2(-1)^\ell x^{s-(\ell-1)/2}}{x^{(s-\ell)/2}}$$
$$\times\left(\sum_{v=0}^{s-\ell}\binom{2s+1}{v}\binom{2s+1}{v+\ell}x^v\right)\left(\sum_{v=0}^{2s}\binom{2s+v}{v}\binom{2s+\ell}{\ell+v}(x-1)^{2s-v}\right),$$

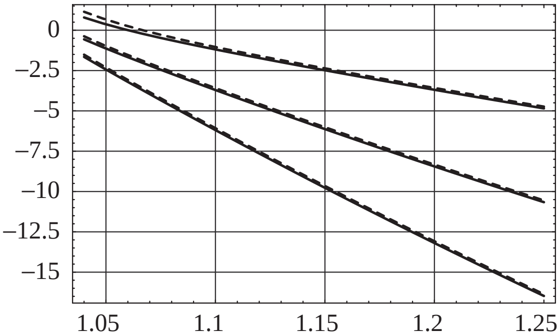

Fig. 2. Log_{10} of the values $L_{n,2}^{(1,2)}(\mathscr{E}_\varrho)$ (solid lines) and their bounds given by (4.10) (dashed lines) for $n = 10, 20, 30$.

where $\sum'$ indicates that the first term of the sum is taken with factor $\frac{1}{2}$. Putting

$$V_\ell(x) = \left(\sum_{v=0}^{s-\ell} \binom{2s+1}{v}\binom{2s+1}{v+\ell} x^v\right)\left(\sum_{v=0}^{2s}\binom{2s+v}{v}\binom{2s+\ell}{\ell+v}(x-1)^{2s-v}\right)$$

and $P_s(x) = V_0(x) + 2\sum_{\ell=1}^{s}(-1)^\ell V_\ell(x)$, we obtain

$$H(x) = 2^{2s+1}\pi x^{(s+1)/2}\frac{P_s(x)}{(x-1)^{4s+1}}.$$

Note that $\deg P_s(x) = 3s$.

Finally, (4.11) reduces to

$$L_{n,s}^{(p,q)}(\mathscr{E}_\varrho) \leqslant 2\pi\sqrt{C_{k,m}(a_1)\frac{P_s(x)}{(x-1)^{4s+1}}}, \quad x = \varrho^{4(n+1)}. \qquad \square$$

Remark 4.4. The polynomials $P_s(x)$ in (4.10), for $s = 0, 1, 2, 3$, are

$$P_0(x) = 1, \quad P_1(x) = 1 - 5x + 19x^2 + 9x^3,$$

$$P_2(x) = 1 - 9x + 36x^2 + 16x^3 + 1251x^4 + 1125x^5 + 100x^6,$$

$$P_3(x) = 1 - 13x + 78x^2 - 286x^3 + 1904x^4 + 32964x^5 + 150578x^6 + 148862x^7 + 34251x^8 + 1225x^9.$$

In Fig. 2 we give graphs of $\varrho \mapsto \log_{10}(L_{n,s}^{(p,q)}(\mathscr{E}_\varrho))$ and their bounds (4.10), for $n = 10, 20$, and 30, when $s = 2$, $k = 1$, $m = 2$. As we can see, the bounds (4.10) are very precise especially for larger values of n and ϱ.

A similar analysis can be done for the Gauss–Turán–Radau formula (3.9).

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